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Three ways to solve the Poisson equation on a sphere with Gaussian forcing

John P. Boyd *, Cheng Zhou

Department of Atmospheric, Oceanic and Space Science, University of Michigan, 2455 Hayward Avenue, Ann Arbor, MI 48109-2143, United States

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ABSTRACT

Motivated by the needs of vortex methods, we describe three different exact or approximate solutions to the Poisson equation on the surface of a sphere when the forcing is a Gaussian of the three-dimensional distance, $\nabla^2\psi = \exp(-2\epsilon^2(1-\cos(\theta)) - C^{Gauss}(\epsilon)$. (More precisely, the forcing is a Gaussian minus the "Gauss constraint constant", C^{Gauss}; this subtraction is necessary because ψ is bounded, for any type of forcing, only if the integral of the forcing over the sphere is zero [Y. Kimura, H. Okamoto, Vortex on a sphere, J. Phys. Soc. Jpn. 56 (1987) 4203–4206; D.G. Dritschel, Contour dynamics/surgery on the sphere, J. Comput. Phys. 79 (1988) 477–483]. The Legendre polynomial series is simple and yields the exact value of the Gauss constraint constant, but converges slowly for large ϵ. The analytic solution involves nothing more exotic than the exponential integral, but all four terms are singular at one or the other pole, cancelling in pairs so that ψ is everywhere nice. The method of matched asymptotic expansions yields simpler, uniformly valid approximations as series of inverse even powers of ϵ that converge very rapidly for the large values of ϵ ($\epsilon > 40$) appropriate for geophysical vortex computations. The series converges to a nonzero $O(\exp(-4\epsilon^2))$ error everywhere except at the south pole where it diverges linearly with order instead of the usual factorial order.

1. Introduction

Vortex methods have been a popular computational strategy in fluid mechanics even before electronic computers. Rosenhead extended Lord Rayleigh's linearized theory of the roll-up of a vortex sheet (Kelvin–Helmholtz instability) into the nonlinear regime by approximating the vortex sheet by a row of point vortices, which are idealizations in which all the vorticity is concentrated into a point. To compute the streamfunction, he analytically solved the Poisson equation for a forcing that was a Dirac delta function. Superimposing the solutions for each point vortex yielded a coupled system of ordinary differential equations in time that described the changing coordinates of each point vortex as a result of their mutual interaction. Armed with nothing more than a mechanical calculator, he generated numerical solutions with eight vortices [30].

Later studies have used thousands of point vortices to confirm Rosenhead's qualitative picture [21,22], but there is a fundamental difficulty: a smooth field of vorticity is approximated by a collection of delta function spikes. More recent work has therefore shifted to vortex blobs; a popular species of blob is the "Gaussian vortex" in which the vorticity falls as an exponential of the square of distance from the center of the vortex [31,3]. Point vortex methods have been applied to flow on the surface of a sphere [19,2,14,17,15,16,13,18,20,24–29,32–37,39]. However, extensions of Gaussian vortex methods to the sphere, as appropriate for adaptive modeling of atmospheric and oceanic flows, are handicapped by the lack of explicit solutions for the Poisson equation with Gaussian forcing.

* Corresponding author. Tel.: +1 734 764 3338.
E-mail address: jpboyd@umich.edu (J.P. Boyd).

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Here, we give three different solutions to the Poisson equation for a Gaussian centered on the north pole with the streamfunction dependent on colatitude θ only:

$$\nabla^2\psi = \exp(-2\epsilon^2(1-\cos(\theta))) - C^{Gauss}(\epsilon) \tag{1}$$

where ϵ is a positive constant and $C^{Gauss}(\epsilon)$ is the "Gauss Constraint constant" (Fig. 1). There is no loss of generality in restricting our quest to a Gaussian centered on the north pole because the solution for a Gaussian centered at an arbitrary point on the sphere can be obtained by a coordinate rotation given explicitly in Section 5 below.

Although our personal motivation is vortex methods for fluids, the Poisson equation arises in many fields of science and engineering. Our method is completely general in that $\nabla^2\psi = f$ can be solved by expanding the forcing f in a Gaussian RBF series; the streamfunction is then a series of translated copies of the solution for a single Gaussian that is derived, in three different forms, below. Generalizations of this "method of fundamental solutions" (though not the Poisson equation on a sphere) are ably reviewed in [23].

There are multiple ways to measure distance on the sphere. Our forcing is Gaussian in the three-dimensional Euclidean distance on the sphere as used by [4,10–12] instead of the "geodesic" distance used in some other studies. For small θ (near the north pole),

$$\exp(-2\epsilon^2(1-\cos(\theta))) - C^{Gauss}(\epsilon) \approx \exp(-\epsilon^2\theta^2), \quad \theta \ll 1 \tag{2}$$

This shows explicitly that the forcing is indeed a Gaussian and the e-folding scale of its decay is $1/\epsilon$ (where the "e-folding scale" is the distance from the center to where ψ has fallen to $\exp(-1)$ of its maximum).

We claim no novelty for the solution in the form of an infinite series of Legendre polynomials, but modern computers make it easy to determine how many terms must be retained in the truncation. The Legendre series also yields the exact value of the Gauss Constraint constant which must be added to the Gaussian forcing so that the vorticity, averaged over the sphere, is zero. The Gauss Constraint is a necessary condition for the Poisson solution ψ to be bounded.

By introducing a new symbol v for $d\psi/dx$, we reduce the order of the differential equation and derive the exact solution. Even though exact solution is smooth everywhere, the analytic expression has the flaw that two terms are singular at the north pole and two other terms are singular at the south pole; the solution is smooth only because of exact cancellation of these singularities. Furthermore, the exact solution is too complicated to allow for easy interpretation.

Therefore, we solve the problem a third way by using matched asymptotic expansions for large ϵ. This is reasonable because for vortex blob methods, we are interested only in *very* large ϵ. For example, the maximum tangential velocities of a Gaussian vortex occur at $1/\sqrt{2}$ of the e-folding scale. For a hurricane, a radius-of-maximum-tangential-winds of 100 km is not uncommon, implying $\epsilon \approx 40$. In adaptive modeling of the atmosphere, we would in reality wish to represent a hurricane by a flock of overlapping vortices of much smaller size than the storm rather than by a single vortex. To match the resolution of current operational forecasting models, we need $\epsilon > 1000$.

The perturbation series is an expansion in inverse powers of the *square* of ϵ, so the series converges extremely fast for geophysical applications.

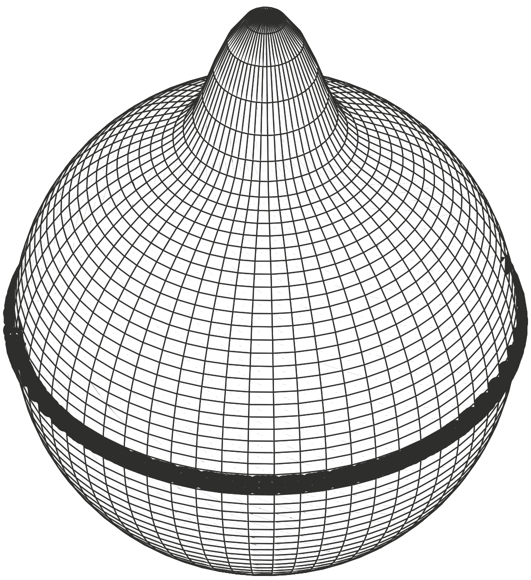

Fig. 1. Schematic of a Gaussian forcing centered on the north pole, here for $\epsilon = 5$. The thick black curve is the equator.

2. Solution by spherical harmonics/Legendre polynomials

2.1. General solution

The spherical harmonics are defined to be the eigenfunctions of the Laplace operator on the surface of a sphere. Nineteenth century theory showed that these are a complete set of orthogonal basis functions for the sphere. It is easy to solve the Poisson equation on the sphere for any forcing ζ merely by expanding both ζ and ψ as spherical harmonic series.

Theorem 2.1. *Suppose a function ζ has the spherical harmonic series*

$$\zeta = \sum_{n=0}^{N} \sum_{m=-n}^{n} \zeta_{mn} Y_n^m \tag{3}$$

where

$$\zeta_{mn} = \int_S \zeta(\lambda,\theta) Y_n^m(\lambda,\theta) \sin(\theta) d\theta\, d\lambda \tag{4}$$

If the "Gauss constraint",

$$\int_S \zeta(\lambda,\theta) \sin(\theta) d\theta\, d\lambda = 0 \quad \leftrightarrow \quad \zeta_{00} = 0, \tag{5}$$

is satisfied, then the Poisson equation on a sphere,

$$\nabla^2 \psi = \zeta, \tag{6}$$

has the general series solution

$$\psi(\lambda,\theta) = \sum_{n=1}^{N} \sum_{m=-n}^{n} \left(-\frac{1}{n(n+1)}\right) \zeta_{mn} Y_n^m(\theta,\lambda) \tag{7}$$

Proof. The eigenequation of the spherical harmonics is

$$\nabla^2 Y_n^m = -n(n+1) Y_n^m \tag{8}$$

Substituting spherical harmonic series for both ζ and ψ, applying the eigenrelation, and matching spherical harmonic terms gives the theorem.

The "Gauss constraint" is necessary because the sole $n = 0$ term, $Y_0^0(\lambda,\theta) = 1$ has an eigenvalue of *zero*. To avoid dividing by zero, we must omit the $n = 0$ term. This omission requires that $\zeta_{00} = 0$. □

2.2. Gaussian vortex centered on the north pole

When ζ is *axisymmetric*, that is, varies only with θ, the Poisson equation degenerates into an ordinary differential equation in colatitude. The spherical harmonic series degenerates into a series of Legendre polynomials.

The "Gauss constraint" makes it impossible to find a bounded solution for a purely Gaussian forcing, so the problem must be one of a Gaussian minus a constant C^{Gauss} which is chosen so that the zeroth coefficient of the Legendre series of the total vorticity ζ is zero, that is,

$$\int_0^{\pi} \{\exp(-2\epsilon^2(1-\cos(\theta))) - C^{Gauss}\} \sin(\theta) d\theta = 0 \tag{9}$$

Theorem 2.2. *The solution to*

$$\nabla^2 \psi = \exp(-2\epsilon^2(1-\cos(\theta))) - C^{Gauss}(\epsilon) \tag{10}$$

where the "Gauss constraint constant" C^{Gauss} is

$$C^{Gauss} = \frac{1}{4\epsilon^2}\{1 - \exp(-4\epsilon^2)\} \tag{11}$$

is

$$\psi = \sum_{n=1}^{\infty} \left(-\frac{1}{n(n+1)}\right) \frac{(2n+1)}{2} \frac{\sqrt{\pi}}{\epsilon} \exp(-2\epsilon^2) I_{n+1/2}(2\epsilon^2) P_n(\cos(\theta)) \tag{12}$$

where the $P_n(x)$ are the usual unnormalized Legendre polynomials and $I_{n+1/2}$ are the usual modified spherical Bessel functions.

Proof. Follows from the previous theorem and the Legendre expansion of a Gaussian in [4]. □

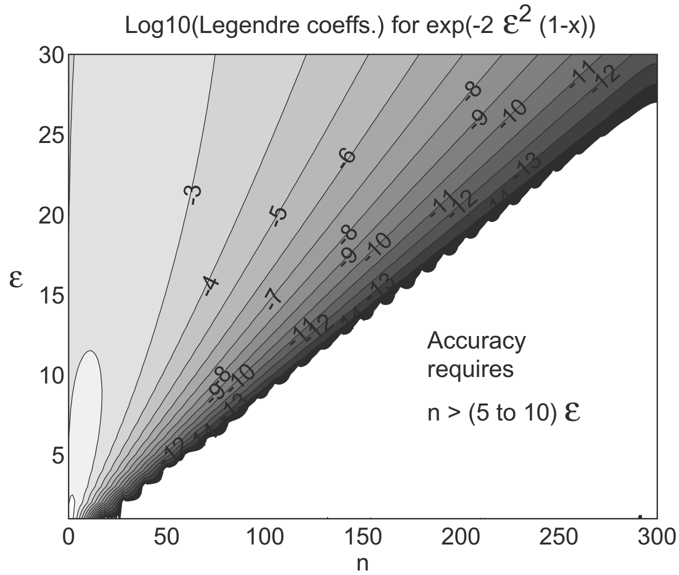

Fig. 2. Isolines of the base-10 logarithm of the absolute value of the Legendre coefficients in $x = \cos(\theta)$ for $\zeta(\theta) = \exp(-2\epsilon^2(1-x)) - C^{Gauss}(\epsilon)$ versus degree and ϵ. Because $|P_n(x)| \leqslant 1$ for all $x \in [-1, 1]$, the error in truncating after the n-th term is only slightly larger than the n-th coefficient according to the "Last Coefficient Rule-of-Thumb" in Chapter 2, Section 12 of [6].

The I-Bessel functions of half-integral order are so-called "spherical" Bessel functions: each $I_{n+1/2}(x)$ is the product of an exponential with a polynomial of degree $(n+1)$ in $1/x$ [1]. One can numerically compute N coefficients in $O(N)$ operations by using the usual three-term recurrence relations for Bessel functions. The Legendre series converges "supergeometrically" in the language of [6] because both ζ and ψ are entire functions, analytic everywhere in the complex θ-plane. Furthermore, the Gauss constraint falls out naturally.

These virtues are offset by the unfortunate fact that the rate of convergence, though always exponential, is very slow for large ϵ as illustrated in Fig. 2. Full machine precision requires a Legendre truncation of roughly $N \approx 10\epsilon$. Since vortex blob applications typically require very large ϵ, e. g., $\epsilon > 1000$, the Legendre series requires thousands of terms even if we accept an accuracy of only a couple of decimal places. Thus, for large ϵ, something better is needed.

3. Exact solution

When the Gaussian vortex is centered on the north pole, the solution is independent of longitude and the Poisson equation reduces to an ordinary differential equation in colatitude. The boundary conditions are that the solution ψ is analytic everywhere include the poles. It is convenient to eliminate the trigonometric functions by using the coordinate

$$x = \cos(\theta) \tag{13}$$

The differential operator becomes the algebraic form of the operator that appears in the eigenequation of the Legendre polynomials:

$$(1-x^2)\psi_{xx} - 2x\psi_x = \exp(-2\epsilon^2(1-x)) - C^{Gauss} \tag{14}$$

Because the unknown does not appear except in differentiated form, this *second* order equation can be converted to a *first* order equation by defining the new unknown

$$v \equiv \psi_x \tag{15}$$

The inhomogeneous, linear first order equation is

$$v_x - \frac{2x}{1-x^2} v = \frac{1}{1-x^2}\{\exp(-2\epsilon^2(1-x)) - C^{Gauss}\} \tag{16}$$

This can be solved by the usual formula for an ODE of the form

$$v_x + p(x)v = f(x) \tag{17}$$

This yields

$$v(x) = A\exp(Q(x)) + \exp(Q(x))\int^x \exp(-Q(y))f(y)dy \tag{18}$$

where here

$$p(x) = \frac{-2x}{1-x^2} \tag{19}$$

$$Q(x) = -\int^x p(y)dy = \int^x \frac{2y}{1-y^2}\,dy = -\log(1-x^2) \tag{20}$$

which implies that

$$\exp(Q(x)) = \frac{1}{1-x^2} \tag{21}$$

It follows that

$$\exp(-Q(x))f(x) = \{\exp(-2\epsilon^2(1-x)) - C^{Gauss}\} \tag{22}$$

$$\int^x \exp(-Q(y))f(y)dy = \frac{1}{2\epsilon^2}\exp(-2\epsilon^2(1-x)) - C^{Gauss}x \tag{23}$$

The general solution for v (i. e., $d\psi/dx$) is therefore

$$v = \frac{1}{1-x^2}\left\{A + \frac{1}{2\epsilon^2}\exp(-2\epsilon^2(1-x)) - \frac{1}{4\epsilon^2}(1-\exp(-4\epsilon^2))x\right\} \tag{24}$$

where A is the multiplier of the homogeneous solution; this must be chosen so as to obtain a nonsingular solution. Denote the numerator of $v(x)$ by

$$\mathcal{N}(x) \equiv A + \frac{1}{2\epsilon^2}\exp(-2\epsilon^2(1-x)) - \frac{1}{4\epsilon^2}(1-\exp(-4\epsilon^2))x \tag{25}$$

To eliminate the singularities, $\mathcal{N}(\pm 1) = 0$, which requires

$$A = -\frac{1}{4\epsilon^2}(1+\exp(-4\epsilon^2)) \tag{26}$$

Then

$$v = \frac{1}{2\epsilon^2}\frac{\exp(-2\epsilon^2(1-x))}{1-x^2} - \frac{1}{4\epsilon^2}\frac{1}{1-x} - \frac{1}{4\epsilon^2}\exp(-4\epsilon^2)\frac{1}{1+x} \tag{27}$$

Formally integrating once gives the solution for ψ, omitting an arbitrary additive constant which has no physical meaning,

$$\psi = \frac{1}{4\epsilon^2}\{1-\exp(-4\epsilon^2)\}\log(1-x) - \frac{1}{4\epsilon^2}\exp(-4\epsilon^2)\log\left(\frac{1+x}{1-x}\right) + \frac{1}{4\epsilon^2}E_1(2\epsilon^2[1-x]) + \frac{1}{4\epsilon^2}\exp(-4\epsilon^2)\mathrm{Ei}(2\epsilon^2[1+x])$$

where we have employed the definitions of special functions in [1]:

$$E_1(z) \equiv \int_1^\infty \frac{\exp(-zt)}{t}\,dt = \int_z^\infty \frac{\exp(-y)}{y}\,dy \tag{28}$$

$$\mathrm{Ei}(z) \equiv \gamma + \log(z) + \int_0^z \frac{\exp(t)-1}{t}\,dt = \mathcal{P}\int_{-\infty}^z \frac{\exp(t)}{t}\,dt \tag{29}$$

where $\mathcal{P}$ denotes the principal value. In Matlab, $E_1(x) = \mathrm{expint}(x)$ and $\mathrm{Ei}(x) = -\Re(-\mathrm{expint}(-x))$; the Maple equivalents are $E_1(x) = \mathrm{Ei}(1,x)$ and $\mathrm{Ei}(x) = -\Re(\mathrm{Ei}(1,-x))$.

Using standard software to evaluate the exponential functions can cause difficulties at both poles because there are logarithmic singularities that cancel in pairs. It is therefore best to use the leading terms of power series expansions very close to the poles as catalogued in Table 1.

Table 1
Singularities.

Term	Singularity at north pole ($\theta = 0 \leftrightarrow x = 1$)	Singularity at south pole ($\theta = \pi \leftrightarrow x = -1$)
$\frac{1}{4\epsilon^2}\{1-\exp(-4\epsilon^2)\}\log(1-x)$	$\frac{1}{2\epsilon^2}\{1-\exp(-4\epsilon^2)\}\log((\theta))$	$O(\epsilon^{-2})$; nonsingular
$\frac{1}{4\epsilon^2}E_1(2\epsilon^2[1-x])$	$-\frac{1}{2\epsilon^2}\log(\theta)$	$O(\exp(-4\epsilon^2))$; nonsingular
$\frac{1}{4\epsilon^2}\exp(-4\epsilon^2)\mathrm{Ei}(2\epsilon^2[1+x])$	O(1); nonsingular	$\exp(-4\epsilon^2)\frac{1}{2\epsilon^2}\log(\theta-\pi)$
$-\frac{1}{4\epsilon^2}\exp(-4\epsilon^2)\log((1+x)/(1-x))$	$\frac{1}{2\epsilon^2}\exp(-4\epsilon^2)\log(\theta)$	$-\frac{1}{2\epsilon^2}\exp(-4\epsilon^2)\log(\theta-\pi)$

One might suppose that it would be possible to simplify the exact solution and avoid difficulties at the south pole at least by deleting terms multiplied by $\exp(-4\epsilon^2)$. This factor is the amplitude of the vortex ζ at the south pole; it is smaller than machine epsilon, 2.2×10^{-16}, for $\epsilon > 3$, much smaller than would be used in any vortex blob on the sphere. However, each of the four terms in the exact solution is logarithmically singular at either the north or south pole or both. (Fortunately, these singularities are "removable"; these singularities in pairs cancel so that the solution is nonsingular everywhere on the sphere as catalogued in Table 1.)

Crowdy independently derived the analytical solution [7].

4. Interpretation: method of matched asymptotic expansions

4.1. Introduction and the outer approximation

One way of simplifying and interpreting the exact solution for large ϵ is to approximately solve the same problem using the method of matched asymptotic expansions. When $\epsilon \gg 1$, the inhomogeneous term $\zeta(\theta)$ is non-negligible only in a small region near the north pole. However, the smaller the region on the sphere, the more nearly this region is *planar*. This implies that the lowest order "inner approximation" around the north pole will be given by the solution of the Poisson equation with Gaussian forcing on an unbounded *plane*.

Far from the north pole, the forcing is exponentially small and may be neglected. It follows that outside of the "inner" polar cap region whose width is $O(1/\epsilon)$, the streamfunction ψ in this "outer" region must satisfy the *Laplace* equation (to within an error $O(\exp(-4\epsilon^2))$. Furthermore, since the forcing and therefore the solution ψ are both axisymmetric (that is, longitude-independent) over the entire globe, it follows that the Laplacian solution must be axisymmetric. The twin constraints of axisymmetry and solving the Laplacian are so strong that the outer solution is uniquely determined to within an arbitrary multiplicative constant. The outer solution is in fact proportional to the known Green's function for the Poisson equation on the sphere, which is the limit $\epsilon \to \infty$ of the solution for a Gaussian forcing and also is the Poisson solution for a point vortex.

This assertion that the outer solution is proportional to that of a point vortex on the sphere can be justified more rigorously by solving Poisson's equation on a spherical domain that excludes a polar cap of $\theta \leqslant \theta^{cap}$ and decomposing the solution into two parts. One part, $\psi^{Laplace}$ solves the Laplace equation with the boundary condition $\psi^{cap}(\theta^{cap}) = q$ where q is a constant. The second part solves the Poisson equation with a forcing which is everywhere smaller than $\exp(-2\epsilon^2)$; because the solution to the Poisson equation is the same order of magnitude as the forcing, it follows that $\psi^{Poisson} \sim O(\exp(-2\epsilon^2))$ and thus is negligible for large ϵ.

The uniqueness of the solution to Laplace's equation implies that $\psi^{Laplace}(\theta; q)$ is unique. The solution $\psi(\theta, \epsilon)$ to the Gaussian-forced Poisson equation over the globe will have some value $\psi(\theta^{cap}, \epsilon)$ at $\theta = \theta^{cap}$. It follows that if the boundary value q for the

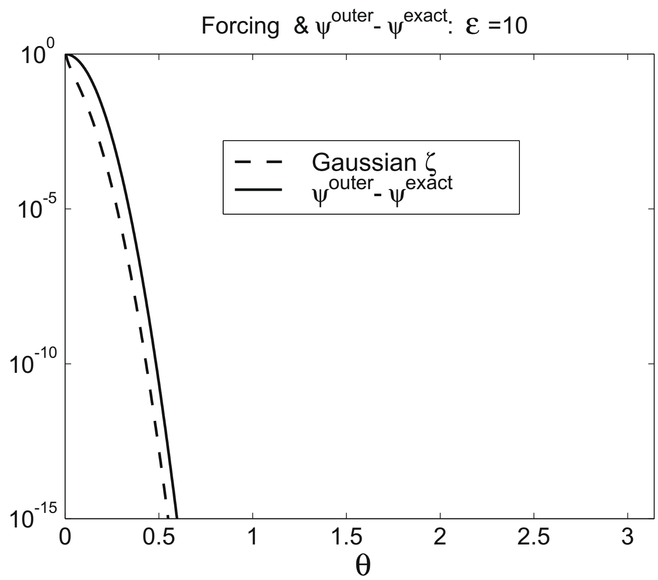

Fig. 3. Decay of the forcing and of the difference between the exact solution and the solution for a point vortex at the north pole ("outer solution") for $\epsilon = 10$. The graphs show that the exact ψ is tending to ψ^{outer} at a *Gaussian* rate, that is, roughly as fast as the forcing ζ is decaying with distance from the north pole. Both curves are below the axis limit, 10^{-15}, for $\theta > 1/2$; we have deliberately kept the graph range as $\theta \in [0, \pi]$, which is the whole globe, to emphasize that the effects of distant vortices or sources can be handled by fast multipole methods or treecodes exactly as in point vortex models.

Laplace solution is chosen to be $q = \psi(\theta^{cap}, \epsilon)$, then the Laplacian solution will approximate the Gaussian-forced Poisson solution for $\theta > \theta^{cap}$ with an error no worse than $O\{\exp(-2\epsilon^2(1-\cos(\theta^{cap})))\}$, an error which falls *exponentially* fast as $\epsilon \to \infty$.

In most applications of matched asymptotics, the inner and outer solutions are both given by infinite power series in the small parameter. Our problem is simpler because the *exponential* rate of decay of the inner approximation to the outer solution, illustrated in Fig. 3, implies that the corrections to the outer solution are smaller than any finite inverse power of ϵ. This implies that to *all orders* in perturbation theory, the outer approximation is the point vortex solution:

$$\psi^{outer}(\theta;\epsilon) \sim q\log(1-x) + C + O(\exp(-2\epsilon^2)) \tag{30}$$

where as before $x \equiv \cos(\theta)$, q is a constant that must be determined by matching to the inner approximation and where C is an arbitrary constant. (The constant arises because the solution to the Poisson equation is only determined to within an arbitrary constant; we will later choose C as convenient for matching.) The statement that the outer approximation is an "all orders" approximation does not imply that it is exact [it is not], but only that the error is *exponentially small* in ϵ. The constant $q(\epsilon)$ must be determined by asymptotic matching to the inner solution.

The exponentially-fast decay to the outer solution is also important for another reason. Point vortex methods on the sphere have been accelerated by using fast multipole methods or related algorithms. Far from the region of nonzero vorticity, the effects of an entire cluster of point vortices can be approximated by a single low order Laurent or power series, thus replacing many terms by a few. Exactly the same strategy can be applied with Gaussian vortices because the "far field" or "outer solution" is the *same* to within *exponentially small error*, that is, equal to that of the point vortex, whenever the Gaussians are narrow compared to the radius of the sphere.

4.2. Shifted coordinate

It is convenient to define a shifted coordinate

$$z = 1 - x \quad \leftrightarrow \quad x = 1 - z \tag{31}$$

so that the new coordinate z is "quasi-radial" in the sense that $z = 0$ at the north pole, the peak of the Gaussian forcing. The differential equation becomes without approximation

$$z(2-z)\psi_{zz} + 2(1-z)\psi_z = \exp(-2\epsilon^2 z) - C^{Gauss} \tag{32}$$

The logarithm in the outer approximation helpfully simplifies to $\log(z)$.

4.3. The inner approximation

Because the Gaussian forcing decays on an $O(\epsilon^2)$ length scale in z, it is convenient to define a rescaled inner coordinate Z, as usual in matched asymptotics, where

$$Z \equiv z\epsilon^2 \quad \leftrightarrow \quad z = Z/\epsilon^2 \tag{33}$$

$$\frac{d}{dz} = \epsilon^2 \frac{d}{dZ} \tag{34}$$

Neglecting terms exponentially small for $\epsilon \gg 1$, as already necessary in the outer approximation, simplifies the Gauss constraint constant to

$$C^{Gauss} \sim \frac{1}{4\epsilon^2} \tag{35}$$

The equation in the inner coordinate is then

$$Z(2 - Z/\epsilon^2)\psi_{ZZ} + 2(1 - Z/\epsilon^2)\psi_Z = \frac{1}{\epsilon^2}\exp(-2Z) - \frac{1}{4}\frac{1}{\epsilon^4} \tag{36}$$

Note that the constant C^{Gauss} is small compared to the maximum value of the Gaussian, and therefore will appear only in the second order problem at $O(1/\epsilon^4)$.

Expand

$$\psi^{inner} = \frac{1}{\epsilon^2}\{\psi^{1,inner} + \epsilon^{-2}\psi^{2,inner} + \ldots\} \tag{37}$$

The lowest order problem is

$$2z\psi^{1,inner}_{ZZ} + 2\psi^{1,inner}_{Z} = \exp(-2Z) \tag{38}$$

The general solution is

$$\psi^{1,inner}(Z) = \frac{1}{4}E_1(2Z) + C_1\log(2Z) + C_2 \tag{39}$$

Because ψ must be nonsingular at the origin and

$$E_1(2Z) \approx -\log(2Z) + \text{nonsingular terms}, \tag{40}$$

we must choose $C_1 = 1/4$. Because the solution to the Poisson equation, which does not contain an undifferentiated ψ, is determined only within an arbitrary additive constant, we can choose C_2 to be whatever we wish. (In particular, the velocities in vortex flow, which are proportional to derivatives of the streamfunction ψ, are independent of C_2.) The simplest choice, made below, is $C_2 = 0$. We obtain

$$\psi^{1,inner} = \frac{1}{4}\{\mathrm{E}_1(2Z) + \log(2Z)\} \tag{41}$$

Now near the north pole,

$$2Z \approx \epsilon^2\theta^2 \equiv r^2 \tag{42}$$

where r is a radial coefficient in a local polar coordinate system, scaled by ϵ. Expressed in terms of r, the lowest order solution on the sphere is identical with the solution to the Poisson equation with Gaussian forcing on an unbounded two-dimensional plane:

$$\psi^{1,inner} = \frac{1}{4}\{E_1(r^2) + \log(r^2)\} \tag{43}$$

Proceeding similarly to fourth order and explicitly inserting the powers of ϵ gives

$$\psi^{4,uniform} \sim \frac{1}{\epsilon^2}\frac{1}{4}\{E_1(2Z) + \log(2Z)\} + \frac{1}{\epsilon^4}\frac{1}{16}\exp(-2Z) + \frac{1}{\epsilon^6}\exp(-2Z)\left\{\frac{1}{64} + \frac{1}{32}Z\right\} + \frac{1}{\epsilon^8}\exp(-2Z)\left\{\frac{1}{128} + \frac{1}{64}Z + \frac{1}{64}Z^2\right\} \tag{44}$$

or rewritten in terms of x

$$\psi^{4,uniform} \sim \frac{1}{\epsilon^2}\frac{1}{4}\{E_1(2\epsilon^2(1-x)) + \log(2\epsilon^2(1-x))\} + \frac{1}{\epsilon^4}\frac{1}{16}\exp(-2\epsilon^2(1-x)) + \frac{1}{\epsilon^6}\exp(-2\epsilon^2(1-x))\left\{\frac{1}{64} + \frac{1}{32}\epsilon^2(1-x)\right\} + \frac{1}{\epsilon^8}\exp(-2\epsilon^2(1-x))\left\{\frac{1}{128} + \frac{1}{64}\epsilon^2(1-x) + \frac{1}{64}\epsilon^4(1-x)^2\right\} \tag{45}$$

In other singular perturbation problems, it is standard to determine the constant $q(\epsilon)$ that is an overall multiplier of the outer solution by asymptotically matching the inner and outer approximations. Then the inner and outer approximations may be combined into a uniformly valid expansion by (i) adding the inner and outer solutions and (ii) subtracting the inner limit of the outer approximation with all operations performed to a given order. For this problem using the coordinate Z, all this is *unnecessary*: the inner approximation to all orders in *uniformly valid* everywhere on the globe, and not merely where $1 - \cos(\theta) \sim O(\epsilon^2)$. The reasons are that:

1. The inner approximation contains a term proportional to the outer approximation, $q\log((1-\cos(\theta)) + C$ where $q = 1/(4\epsilon^2)$ and $C = \log(2\epsilon^2)/(4\epsilon^2)$.
2. All other terms in the inner approximation (including the term in E_1) decay *exponentially* fast as θ increases away from the north pole.

Because the outer approximation is also the inner limit of the outer approximation, the usual process of uniformizing inner and outer expansions merely adds and subtracts the same term, $(1/4\epsilon^2)\log(2\epsilon^2(1-\cos(\theta))$. Consequently, we have replaced the superscript "inner" by "uniform" in Eq. (44).

It is also possible to derive a uniform approximation by attacking the equivalent differential equation in colatitude

$$\frac{\partial^2\psi}{\partial\theta^2} + \cot(\theta)\frac{\partial\psi}{\partial\theta} = \exp(-2\epsilon^2(1-\cos(\theta))) - C^{Gauss} \tag{46}$$

This yields

$$\begin{aligned}\psi^{4,uniform} = {} & \frac{1}{4}\frac{1}{\epsilon^2}\{\mathrm{E}_1(\epsilon^2\theta^2) + \log\{1-\cos(\theta)\}\} + \frac{1}{\epsilon^4}\exp(-\epsilon^2\theta^2)\left(\frac{1}{16} + \frac{1}{48}\epsilon^2\theta^2\right) \\ & + \frac{1}{\epsilon^6}\exp(-\epsilon^2\theta^2)\left(\frac{1}{64} + \frac{1}{64}\epsilon^2\theta^2 + \frac{31}{5760}\epsilon^4\theta^4 + \frac{1}{1152}\epsilon^6\theta^6\right) \\ & + \frac{1}{\epsilon^8}\exp(-\epsilon^2\theta^2)\left(\frac{1}{128} + \frac{1}{128}\epsilon^2\theta^2 + \frac{1}{256}\epsilon^4\theta^4 + \frac{821}{725760}\epsilon^6\theta^6 + \frac{43}{207360}\epsilon^8\theta^8 + \frac{1}{41472}\epsilon^{10}\theta^{10}\right)\end{aligned} \tag{47}$$

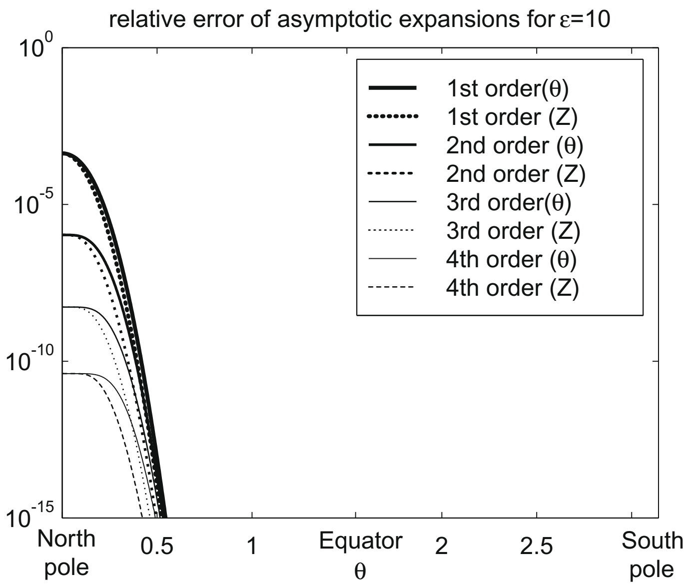

Fig. 4. Errors in the streamfunction ψ for each of the first four orders of the uniformly valid perturbation series when the inverse width of the Gaussian is $\epsilon = 10$. The dashed curves show that the errors in the expansion using the coordinate $Z(\theta)$ are smaller than for the analogous series in θ.

As illustrated in Fig. 4, this is very accurate even for $\epsilon = 10$, which is far smaller than one is likely to use in vortex blob models. However, the series in θ has more terms. Furthermore, the derivation is more laborious because the inner approximation contains a term proportional to $\log(\epsilon^2\theta^2)$, which is a good approximation to the point vortex solution only near the north pole. This requires non-trivial additions, expansions and subtractions to obtain the uniform approximation. In contrast, when $z = 1 - \cos(\theta)$ is the coordinate, the logarithmic term in the inner approximation is proportional to $\log(2\epsilon^2(1 - \cos(\theta))$, which *is* the outer approximation. For this problem, clearly all coordinates are not created equal.

4.4. Interpretation of matched asymptotics

The lowest order uniform perturbative approximation is much simpler than the exact solution and much easier to interpret. In the same way that the Gaussian forcing generalizes from the plane to the sphere as

$$\exp(-r^2) \quad \rightarrow \quad \exp(-2\epsilon^2(1 - \cos(\theta)) \tag{48}$$

$$\frac{1}{4\epsilon^2}\{\mathrm{E}_1(r^2) + \log(r^2)\} \rightarrow \frac{1}{4}\frac{1}{\epsilon^2}\{\mathrm{E}_1(2\epsilon^2(1 - \cos(\theta))) + \log(2\epsilon^2(1 - \cos(\theta)))\} \tag{49}$$

In other words, both the forcing and the logarithm are extended from plane to sphere by $r^2 \rightarrow 2\epsilon^2(1 - \cos(\theta))$.

We have extended the matched asymptotics expansion to fourth order for two minor reasons. The first minor reason is that this problem is a good illustration for a textbook on singular perturbation theory: the solution at each order can be found explicitly by an algebraic manipulation language like Maple, and the problem is both interesting and simple because the outer approximation is a single term to all orders. Further, the exact solution is available for comparison, explicitly displaying the exponentially small, beyond-all-order terms (proportional to $\exp(-4\epsilon^2)$) which make the asymptotic series divergent. It is anomalous, and therefore also interesting, that the inner approximation is uniformly valid over the whole sphere.

There are two reasons for deriving the perturbation series. First, the exact solution is singular at the south pole, and there is no simple way (other than matched asymptotics!) to remove the singularities. In contrast, the perturbation series asymptotes, exponentially fast, to $\log(1 - \cos(\theta))/(4\epsilon^2)$, the point vortex solution, which is not singular at the south pole.

Second, in vortex blob methods, one must sum many Gaussian vortices, centered at different points on the globe, to obtain the complete flow. The summation can be greatly accelerated by treating the *long-range* interactions by fast multipole methods or tree codes exactly as already done for point vortex methods on the sphere by Sakajo [38,9]. Matched asymptotics shows that long-range interactions – the "far field" of a given vortex – are *identical* with those of point vortices except for an exponentially small error. It follows that Sakajo's treecode can be applied to Gaussian blobs without modification. One can still evaluate the near-field influence by using the exact solution, but because this requires the special function Ei as well as E_1, the perturbative approximation may be more efficient for sufficiently large ϵ.

5. Coordinate rotation

The solution for a Gaussian centered at the north pole is a function only of ϵ and $\cos(\theta)$. In a minor change of notation, let us denote this solution by $\psi^{np}(\theta,\epsilon)$. Suppose the forcing vortex is rotated to lie at the points $(\lambda_{center},\theta_{center})$. Then if the evaluation point is $(\lambda_{eval},\theta_{eval})$, the solution to the Gaussian-forced Poisson equation is

$$\psi(\lambda_{eval},\theta_{eval};\lambda_{center},\theta_{center}) = \psi^{np}(\cos(\theta(\lambda_{eval},\theta_{eval};\lambda_{center},\theta_{center}));\epsilon) \tag{50}$$

where

$$\cos(\theta) = \cos(\theta_{center})\cos(\theta_{eval}) + \sin(\theta_{center})\sin(\theta_{eval})\cos(\lambda_{center}-\lambda_{eval}) \tag{51}$$

6. Derivatives

In fluid mechanics, it is very important to evaluate the derivatives of ψ because these are the fluid velocities.

By using the partial fraction decomposition $1/(1-x^2) = (1/2)(1/(1-x)+1/(1+x))$, (27) for $d\psi/dx(v)$ can be rewritten without approximation as

$$\frac{d\psi}{dx} = \frac{1}{4\epsilon^2}\left(\frac{\exp(-2\epsilon^2(1-x))-1}{1-x} + \frac{\exp(-2\epsilon^2(1-x))-\exp(-4\epsilon^2)}{1+x}\right) \tag{52}$$

The poles at $x=\pm1$ are only apparent; the singularities cancel.

The perturbation series is free of the cancelling singularities at $x=-1$ (the south pole). The lowest order uniform approximation exactly reproduces the terms above $1/(1-x)$ in (52). The fourth term, proportional to $\exp(-4\epsilon^2)$, is dropped from the perturbation series because of its exponential smallness. (This term is singular at the south pole, and therefore not small at the south pole but this is cancelled by the singularity in the third term, $\exp(-2\epsilon^2(1-x))/(1+x)$; we can therefore neglect $\exp(-4\epsilon^2)/(1+x)$ over the whole globe provided we modify the third term so as to remove its singularity at the south pole). The third term does indeed lose its singularity because in the perturbation series, its singular denominator, $1/(1+x)$, is replaced by its Taylor expansion about $x=1$:

$$\frac{1}{4\epsilon^2}\frac{\exp(-2\epsilon^2(1-x))}{1+x} \approx \frac{\exp(-2\epsilon^2(1-x))}{4\epsilon^2}\left\{\frac{1}{2}+\frac{1}{4}(1-x)+\frac{1}{8}(1-x)^2+\dots\right\} \tag{53}$$

The derivatives with respect to λ and θ follow by applying the chain rule:

$$\frac{\partial\psi}{\partial\lambda} = \frac{\partial x}{\partial\lambda}\frac{\partial\psi}{\partial x} = \{\sin(\theta_{center})\sin(\theta_{eval})\sin(\lambda_{center}-\lambda_{eval})\}\frac{\partial\psi}{\partial x} \tag{54}$$

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial\psi}{\partial x} = \{-\cos(\theta_{center})\sin(\theta_{eval}) + \sin(\theta_{center})\cos(\theta_{eval})\cos(\lambda_{center}-\lambda_{eval})\}\frac{\partial\psi}{\partial x} \tag{55}$$

To apply a fast multipole or treecode method to sum the far field of the derivative of a Gaussian vortex, it is necessary that the error in approximating $d\psi/dx$ by the *outer* approximation – the derivative of the point vortex – should be less than the desired error tolerance. This error in approximating the derivative by the point vortex derivative is

$$E^{outer} \equiv \frac{d\psi}{dx} + \frac{1}{4\epsilon^2}\frac{1}{1-x} \approx \frac{1}{4\epsilon^2}\frac{\exp(-2\epsilon^2(1-x))}{1-x} \tag{56}$$

This approximation is inaccurate in a small region around $x=1$ where the treecode/multipole methods are inapplicable anyway; this formula is accurate wherever $|E^{outer}| \ll 1$.

7. Divergence and hyperasymptotics in the perturbation series

Singular perturbation series usually diverge factorially; for a fixed, small value of the perturbation parameter ϵ, the error decreases as more terms are added to the expansion up to some optimum order $N_{opt}(\epsilon)$ and then grows factorially as the series truncation order $N\to\infty$. The underlying reason for the divergence is a mathematical crime: An embedded expansion has been used beyond its radius of convergence.

For example, the Stieltjes function

$$S(1/\epsilon) \equiv \int_0^\infty \frac{\exp(-t)}{1+t/\epsilon}dt \sim \sum_{n=0}^{N}(-1)^n n!\frac{1}{\epsilon^n} \tag{57}$$

has an expansion which is derived by replacing $1/(1+t/\epsilon)$ by the geometric series $\sum_{n=0}^{N}(-1)^n t^n/\epsilon^n$. The mathematical crime is that this embedded expansion converges only for $|t|<\epsilon$ because $1/(1+t/\epsilon)$ is infinite at $t=-\epsilon$, but the interval of integration is *infinite*. The reason that the power series is useful anyway is that the integrand has decayed to less than $\exp(-\epsilon)$ at the limit of convergence of the geometric series; one can prove if $N=N_{opt}(\epsilon)=\text{round}(\epsilon)$, the error is approximately $\sqrt{\pi\epsilon/2}\exp(-\epsilon)$ (p. 123 of [5]).

The geometric series is also used illegitimately in our problem, but the expansion of $1/(1+x)$ about $x=-1$ diverges only at the south pole. Because this expansion is in powers of $(x+1)/2$, it is convenient to introduce

$$y \equiv (1-x)/2 \tag{58}$$

By using the identity

$$\frac{1}{1+x} = \frac{1}{2}\frac{1}{1-y} = \frac{1}{2}\sum_{n=0}^{M-1} y^n + \frac{1}{2}\frac{y^M}{1-y} \tag{59}$$

one can show that the error in truncating the perturbation series for $d\psi/dx$ after Nth order is given without approximation as

$$E_N = \frac{1}{8\epsilon^2}\frac{1}{1-y}\{\exp(-4\epsilon^2 y)y^{N-1} - \exp(-4\epsilon^2)\} \tag{60}$$

from which it follows that

$$\lim_{N\to\infty,\text{fixed}y} E_N = -\frac{1}{8\epsilon^2}\exp(-4\epsilon^2)\frac{1}{1-y} \tag{61}$$

Applying l'Hopital's Rule,

$$\lim_{y\to 1,\text{fixed}\,N} E_N = \frac{1}{2}\exp(-4\epsilon^2)\left\{1 - \frac{N}{4\epsilon^2}\right\} \tag{62}$$

The first limit shows the singular perturbation series does not converge to zero error, but does not diverge factorially with N either; instead, for all x except the south pole, it converges to an error $O(\exp(-4\epsilon^2))$. The second limit shows that at the south pole $(x=-1 \leftrightarrow y=1)$, the error does diverge as $N\to\infty$, but the divergence is *linear* in N rather than proportional to $\exp(N\log(N))[N!]$ as is typical. The error is smallest when $N \approx N_{opt}(\epsilon) = \text{round}(4\epsilon^2)$.

These similarities and differences from other asymptotic-but-divergent series makes the Poisson equation on the sphere an interesting example of singular perturbation series. We do not know of a textbook example with multiple length scales ($O(1/\epsilon)$ for the forcing, $O(1)$ for the spherical geometry) that exhibits a similar convergence to nonzero-but-exponentially small error, nor diverges linearly instead of factorially at a point.

For large ϵ, the maximum error in the Nth order perturbation series occurs at $(1-x)\approx 2(N-1)/\epsilon^2$ and is

$$\max_{x\in[-1,1]} |E_N| \sim \frac{\exp(-[N-1])(N-1)^{N-1}}{2^{1+2N}}\frac{1}{\epsilon^{2N}}, \quad N=1,2,\ldots \tag{63}$$

8. Summary

We present three different explicit solutions to Poisson's equation on the surface of a sphere when the forcing ζ is a Gaussian of arbitrary width located at an arbitrary point on the sphere. This can be extended to a general, adaptive Poisson solver by expanding an arbitrary ζ as a series of Gaussian radial basis functions (RBFs) where the RBF "centers" are concentrated only where ζ is non-negligible.

The Legendre series is the best for very wide Gaussians (small ϵ) and yields the exact Gauss constraint constant for all ϵ.

The exact solution is simple, involving nothing more exotic than two variants of the exponential integral.

The matched asymptotic expansion is valuable because the outer solution is the same as for point vortex methods. It is therefore possible to greatly accelerate the RBF/Poisson solver by using fast multipole methods or treecodes exactly as for point vortex methods.

The singular perturbation series diverges linearly, instead of with the usual factorial rate, at the south pole. It converges to a nonzero error of $O(\exp(-4\epsilon^2))$ everywhere else on the globe. The inner approximation is also a spatially uniform approximation because it contains the outer approximation and the other terms decay exponentially fast away from the north pole. Because of these unusual features and its simplicity, this series is an interesting example for a textbook on singular perturbation methods.

For the very large value of ϵ that will be used in geophysical vortex blob models, the lowest order or two of the perturbation series is extremely accurate and simpler than the exact solution.

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